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Pseudo-boundaries in discontinuous two-dimensional maps

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Abstract. Kolmogorov–Arnold–Moser boundaries appear in sufficiently smooth two-dimensional area-preserving maps. When such boundaries are destroyed, they become pseudo-boundaries. We show that pseudo-boundaries can also be found in discontinuous maps. These pseudo-boundaries originate in groups of chains of islands which separate parts of the phase space and need to be crossed in order to move between the different subspaces. Trajectories, however, do not easily cross these chains, but tend to propagate along them. This type of behaviour is demonstrated by using a 'generalized' Fermi map.

The most obvious way to set particles into motion is to subject them to an external field. Yet, particle currents can also be induced by zero-averaged time-dependent forces, provided an asymmetric potential is applied in the system. Recent studies of systems consisting of Brownian particles in periodic, locally asymmetric, potentials have shown that motion may be induced by periodic forcing, non-random noises or due to an oscillatory change in the potential profile [1]. Such rectification processes are interesting because they can lead to the generation of new microscopic pumping techniques. They may also provide some insight into the mechanism of protein motors.

Another example for a net directed motion of particles in periodic (both in time and space) potentials is found in a model proposed by Jarić and Sundaram [2]. In this model a classical particle moves freely between an infinite set of equally spaced potential barriers of infinitesimal width, whose position and height oscillate periodically. Once the particle collides with a barrier, it either crosses or reflects from the barrier, depending whether or not its kinetic energy is larger than the potential height at the moment of impact (interaction between the particle and the barrier is instantaneous). The special feature of this system (unlike these in [1]) is its chaotic deterministic nature. The velocity of the particle and the phase of the barrier at the nth impact ($n = 1, 2, \ldots$) can be computed by a two-dimensional area-preserving map. This map defines the system's two-dimensional phase space and the dynamics in this phase space. As is frequently done for chaotic systems, the dynamics of the particle was studied by investigating the corresponding trajectories in the phase space [3, 4].

Points in the phase space can be divided into two classes, each corresponding to either crossing or reflection from the barrier (this follows from the instantaneous character of interactions between the barrier and particle). On the boundaries between regions containing points of different classes the map is discontinuous (see figure 1). In this paper we report and

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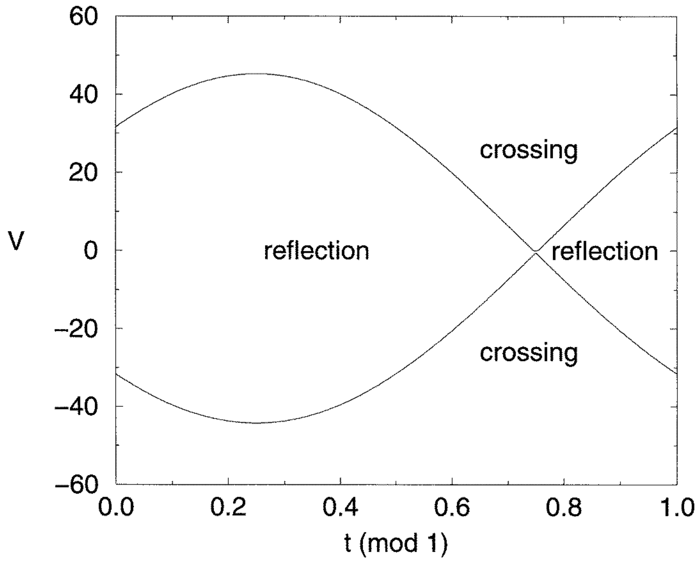

Figure 1. Regions in the $(t - V)$ phase space of points mapped by the submaps of crossing and reflection, for a barrier mean height $H_0 = 500$. The map is discontinuous over the curves separating the different regions.

qualitatively explain an interesting phenomenon observed while studying this discontinuous map. Pictures of the phase space constructed (by plotting several trajectories originating from different initial conditions) for different values of the map's parameters, displayed two types of dynamics in the phase space. Most of the phase space consists of a *stochastic sea*, which a typical trajectory covers ergodically. Some initial conditions, on the other hand, led to *regular* dynamics over Kolmogorov–Arnold–Moser (KAM) curves. Considering the diffusion of trajectories in the stochastic regimes, the existence of these KAM curves is very important. If they exist, they bound different parts of the stochastic sea, that is, prevent trajectories from leaving or entering them. KAM curves usually appear in two forms: closed curves form small isolated islands in the stochastic sea. Open (non-closed) curves, which are 'stretched along the stochastic sea', disconnect the stochastic sea into separate, inaccessible seas. According to the KAM theorem [5–7], KAM curves do not appear in discontinuous maps as is ours†. To be more accurate: a KAM curve cannot intersect a discontinuity line of the map. Therefore, in our map, small islands do appear provided that they lie in a region where the map is smooth, i.e. on one side of the discontinuity lines shown in figure 1 (they appear around *locally* stable periodic points). Open KAM curves, which intersect a discontinuity line, do not exist. The name KAM curve is used throughout this paper to denote the forbidden open curves.

Since KAM curves do not exist, the stochastic sea is a single connected unit with only a few islands excluded. The motion over a connected (invariant) component of the phase space is ergodic and therefore a trajectory starting at almost any point in the stochastic sea, will eventually explore this entire region. Numerical observations have shown that the number of iterations needed to explore completely the stochastic sea varies, within more than three orders of magnitude, for different values of the map's parameters. When this number was

† For two-dimensional nearly integrable maps, the continuity of the first derivative is a necessary condition for the existence of KAM curves, while the continuity of the second derivative is a sufficient one [8].

large, the stochastic sea seemed to be divided into smaller parts in which trajectories travelled for long time intervals before moving from one to the other. The boundaries between the different parts are called *pseudo* (*partial*) boundaries. Their appearance does not violate the ergodicity hypothesis, it simply implies that it is valid only for timescales much larger than the characteristic escape time from each pseudo-bounded part of the stochastic sea.

Each KAM boundary is destroyed when the map's control parameter exceeds some critical value. At the critical point an infinite number of gaps are formed in the torus and it turns into a Cantor set (hence called a *cantorus*) [9, 10]. As we continue to change the control parameter, the flux across the cantorus increases from its zero value at the critical point†. Thus, close to the critical point, the flux across a cantorus is low and cantorus serves as a pseudo-boundary. Cantori, like KAM tori, do not appear in discontinuous maps (an exception are piecewise linear maps [11]). In this paper we show that pseudo-boundaries are found in discontinuous maps due to other reasons. They are formed by chains of islands surrounding different areas of the phase space. In order to leave these areas, trajectories need to diffuse *across* the chains, which usually appear in groups. However, the motion in the vicinity of the chains, tends to be *along* the chains, in a way that resembles the regular motion over the chains.

In smooth maps, cantori and chains of islands are deeply related. For each torus, cantorus or chain of islands, there is a corresponding *winding number* (for the definition of winding numbers see, for example, [12]). Tori and cantori have irrational winding numbers while chains of islands have rational ones. Green [13] has shown that close to any torus or cantorus, there are infinitely many chains of islands whose winding numbers are successive finite truncations of the continued fraction representing the irrational winding number of the torus or cantorus. It has also been shown [14, 15] that the flux across a cantorus is the limit of the flux across its approximating chains of islands. This last result means that chains of islands which approximate cantori that serve as pseudo-boundaries (i.e. which the flux across them is low) are pseudo-boundaries themselves. For discontinuous maps, this picture does not hold. We do not attempt to explain why chains (which are actually long periodic orbits whose surrounding islands have chain-like pattern in the phase space) exist in certain locations in the phase space. We just want to demonstrate that once chains appear they can form a pseudo-boundary for the diffusion in the phase space.

We consider a classical particle of unit mass, moving in one dimension between equidistant potential barriers, each of an infinitesimal width and finite height, $H(t)$. The barriers oscillate with the same frequency and in phase. Their velocity is given by $v(t) = v_b g(t)$, while $H(t) = H_0[1 + g(t)]$, where $g(t) = \sin(2\pi t)$. The particle moves freely between the barriers, occasionally colliding with them. At each impact it can either cross or be reflected from the barrier. It crosses the barrier if its kinetic energy in the reference frame of the barrier exceeds the height of the potential barrier at the moment of impact, i.e. $|V_n - v(t_n)| > \sqrt{2H(t_n)}$, where t_n and V_n are, respectively, the time of the nth impact and the velocity of the particle before that impact. If the particle crosses the barrier, its velocity is not changed. If it is reflected, on the other hand, its velocity reverses is sign and changes by twice the velocity of the barrier. In order to calculate the moment of the next impact, one needs to consider the motion of both the particle and the barriers. We made the approximation that the distance travelled between two consecutive collisions is constant and equals to the distance between two barriers, L, as if the barriers do not change their position. A similar approach was used in [16] for the Fermi accelerator model

† The flux across a curve is the fraction of trajectories (uniformly distributed on one of the sides of the curve) which cross the curve in a time unit (e.g. per iteration). This flux is proportional to the total area from which trajectories are mapped form one side of the curve to the other.

(a ball bouncing between two walls [17]). All this can be summarized by the following two-dimensional map:

$$V_{n+1} = \begin{cases} V_n & |V_n - v(t_n)| > \sqrt{2H(t_n)} \text{ (crossing submap)} \\ -V_n + 2v(t_n) & |V_n - v(t_n)| \leqslant \sqrt{2H(t_n)} \text{ (reflection submap)} \end{cases} \tag{1}$$

$$t_{n+1} = t_n + \frac{L}{|V_{n+1}|} \pmod 1. \tag{2}$$

We note that: (1) only time modulo period (= 1) of $g(t)$, which we denote as phase, is relevant. (2) The approximation made in assuming that the barriers do not change their position, leads to a non-physical description of the dynamics when the velocity of the particle is comparable with or smaller than $v(t)$. In particular, although the particle is reflected backwards, the reflection submap may not reverse the sign of V_n. In order to overcome this non-physical situation, we replace V_{n+1} with $-V_{n+1}$ if it has the same sign as V_n.

We chose the mean height of a barrier, H_0, to be the control parameter, while we set the other parameters $L = 100$ and $v_b = \frac{1}{2}$. For the special case $H_0 = \infty$, barrier crossing is impossible and the map reduces to the Fermi map†. For finite H_0, the map is discontinuous. Figure 1 shows the discontinuity lines in the $(V - t)$ phase space which separate the regions mapped by the different submaps of crossing and reflection (for $H_0 = 500$). The map is area-preserving since both submaps are area preserving and since their ranges do not overlap. For $V > V_{\max} \equiv 2\sqrt{H_0} + v_b$ and $V < V_{\min} \equiv -2\sqrt{H_0} + v_b$, points in the $(V - t)$ phase space are mapped only by the crossing submap (the kinetic energy is sufficiently large to cross the barriers at any phase), hence the motion in this part of the phase space is over the lines $V = V_0 =$ constant. The stochastic sea lies between $V_{\min}$ and $V_{\max}$, including several islands. As already explained, a picture of it is made by starting at almost any one of its points, iterating the map a sufficiently large number of times and plotting the resulting trajectories, $\{(t_n, V_n); n = 0, 1, 2 \ldots\}$.

We performed this procedure for various values of H_0. For H_0 values below 1000, it took less than 10^3 iterations for a typical trajectory to visit almost any region of the stochastic sea. After 10^6–10^7 iterations we could not observe an improvement in the resolution of the plots of the phase space. As expected, apart from some embedded chains of islands, the trajectory wandered in the entire region between $V_{\min}$ and $V_{\max}$. For $H_0 > 1000$, the situation was quite different. Figure 2 depicts the velocities V_n and phases t_n of the first 2×10^{10} collisions between the particle and the barriers. The velocities appear to remain smaller than 63, while the stochastic sea ranges between $V_{\min} = -99.5$ and $V_{\max} = 100.5$.

A pseudo-boundary exists around $|V| \sim 63$. A closer look at that area of the phase space reveals a few adjacent chains of islands, located one above the other, whose shapes are similar to the boundary of the area plotted in figure 2. One of these chains is shown in figure 3. It is composed of two velocity branches, in the upper ($V > 0$) and lower ($V < 0$) halves of the phase space, between which the motion alternates. The effect of this chain on the motion close to it is depicted in the inset of figure 3, showing an enlargement of a small segment of the chain. A narrow stochastic layer appears below the chain (above it, if the negative velocity branch is considered). We found that trajectories can move over this narrow layer for thousands of iterations before leaving it, usually falling back into the stochastic sea. (In several simulations we were able to find trajectories which moved $\sim 10^5$ successive iterations over this specific layer.) Two features resemble the behaviour close to

† The standard Fermi map describes a ball bouncing between a fixed and an oscillating wall, while for $H_0 = \infty$ our map describes a motion between two oscillating walls.

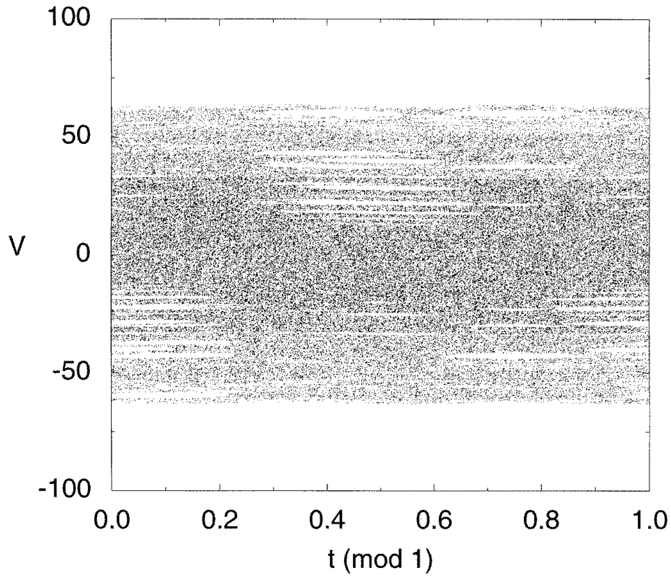

Figure 2. The subspace explored after 2×10^{10} iterations of the map for a barrier mean height $H_0 = 2500$. Each point denotes the velocity of the particle V_n and the phase of the barrier, t_n, at the nth collision.

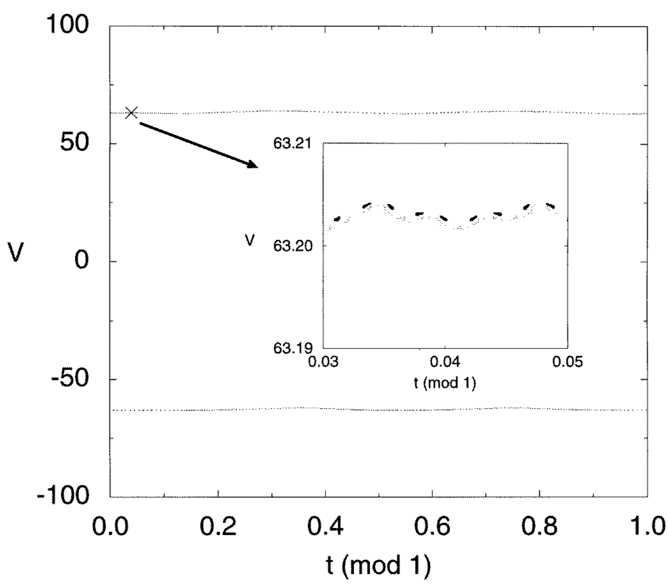

Figure 3. A chain which appears in phase space around $|V| \sim 63$ for a barrier mean height $H_0 = 2500$. Its shape resembles the boundary of the area shown in figure 2. The inset is a magnification of a small part of the chain, showing few of its islands. A narrow stochastic layer is located just below the chain which serves as a pseudo-boundary. Trajectories that reach the stochastic layer, may move over it for thousands of iterations before either falling back into the stochastic sea (the usual case) or crossing the chain (the rare scenario).

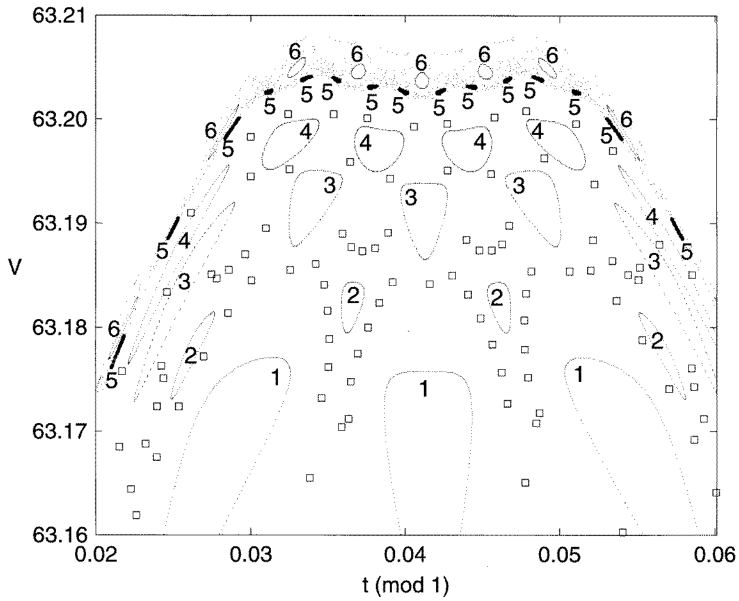

Figure 4. A few adjacent chains of islands found in the phase space for a barrier mean height $H_0 = 2500$, forming together a pseudo-boundary. The numbers inside the islands label the different chains. The number of islands in the chains labelled 1–6, in the upper (lower) half of the phase space, are respectively: 27 (33), 37 (47), 49 (59), 60 (72), 420 (466), 157 (164). The dots and open squares belong to two stochastic trajectories wandering, respectively, above and below the chain labelled 5. (Only a small part of the positive velocity branches of the chains is shown here.)

a KAM torus or a cantorus: (1) the motion is restricted to one side of the boundary (the chain), and (2) the motion is correlated to the boundary and only gradually deviates from it. As for a cantorus, if the chain is long and narrow with only small gaps between the islands, there are only small areas from which a trajectory can jump from one of its sides to the other.

Figure 3 demonstrates the effect of a single long chain with islands located close to each other. The stochastic motion near such a chain is highly correlated. It, roughly, follows the regular motion over the chain and only slowly deviates from it. But when a pseudo-boundary appears, we usually observe a group of several chains, lying closely one above the other, and not only a single chain. Trajectories need to diffuse across all these chains in order to move to another subspace. However, due to the character of the motion near the chains, they tend to move along them. Suppose a trajectory enters between a few chains, close to one of them. It propagates along the chain, while gradually moving away. If there are a few nearby chains, then while it moves away from one chain it gets closer to another, along which it propagates for subsequent iterations. This is an over-simplified picture since it is not always possible to relate the motion to a particular chain at each instance, yet, it qualitatively explains the character of the motion in this 'band of chains'. The trajectory can be trapped between the chains for many iterations. If it succeeds in crossing all of them, it escapes. Usually, however, after spending some time wandering between the chains, it 'falls back' into the stochastic sea. Figure 4 depicts six chains and two stochastic trajectories. Note that while one of the trajectories (open squares) travels between the chains labelled 1–5, the other trajectory (dots) is restricted to the area above the chain labelled 5. Indeed, since the flux across this chain is very small, compared with the flux across the other chains (probably because of the fact that it consists of relatively many islands with small gaps between them), the chain labelled 5 (also shown in the inset of figure 3) serves as the natural limit for both sides of the pseudo-boundary.

In conclusion, we have shown that long and narrow chains of islands may serve as pseudo-boundaries in discontinuous two-dimensional maps. If a group of several adjacent chains surround a certain subspace, they all need to be crossed in order to move to a different subspace. However, in the vicinity of the chains, trajectories which stochastically wander between the islands tend to propagate along the chains, slowly moving from one chain to the other. Hence, the chains are not easily crossed and they form a pseudo-boundary.

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